

A REMARK ON THE GENERIC VANISHING OF KOSZUL COHOMOLOGY

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ABSTRACT. We give a sufficient condition to study the vanishing of certain Koszul cohomology groups for general pairs $(X, L) \in W_{g,d}^r$ by induction. As an application, we show that to prove the Maximal Rank Conjecture (for quadrics), it suffices to check all cases with the Brill-Noether number $\rho = 0$.

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INTRODUCTION

Let L be a base point free g_d^r on a smooth curve X , the Koszul cohomology group $K_{p,q}(X, L)$ is the cohomology of the Koszul complex at (p, q) -spot

$$\longrightarrow \wedge^{p+1} H^0(L) \otimes H^0(L^{q-1}) \xrightarrow{d_{p+1,q-1}} \wedge^p H^0(L) \otimes H^0(L^q) \xrightarrow{d_{p,q}} \wedge^{p-1} H^0(L) \otimes H^0(L^{q+1}) \longrightarrow$$

where

$$d_{p,q}(v_1 \wedge \dots \wedge v_p \otimes \sigma) = \sum_i (-1)^i v_1 \wedge \dots \wedge \widehat{v_i} \wedge \dots \wedge v_p \otimes v_i \sigma.$$

Koszul cohomology groups $K_{p,q}(X, L)$ completely determine the shape of a minimal free resolution of the section ring

$$R = R(X, L) = \bigoplus_{k \geq 0} H^0(X, L^k).$$

and therefore carry enormous amount of information of the extrinsic geometry of X .

In this paper, we are interested in Green's question [10].

Question 0.1. What do the $K_{p,q}(X, L)$ look like for (X, L) general in $\mathcal{W}_{g,d}^r$ (i.e. X is a general curve of genus g and L is a general g_d^r on X)?

The following facts are well known (c.f. [10], [12]) for general $(X, L) \in \mathcal{W}_{g,d}^r$.

- (1) We have the following picture of $k_{p,q} = \dim_{\mathbb{C}} K_{p,q}(X, L)$ (The numbers $k_{p,q}$ in the table are undetermined.):

TABLE 1.

0	$h^1(L)$	0	0	0	0
0	ρ	$k_{r-2,2}$	$k_{2,2}$	$k_{1,2}$	$k_{0,2}$
0	$k_{r-1,1}$	$k_{r-2,1}$	$k_{2,1}$	$k_{1,1}$	0
0	0	0	0	0	1

(2)

$$\begin{aligned}
& k_{p,1} - k_{p-1,2} = \chi(\text{Koszul complex}) \\
& = \binom{r+1}{p}(g-d+r) - \binom{r+1}{p+1}g + \binom{r-1}{p}d + \binom{r}{p+1}(g-1).
\end{aligned}$$

Question 0.1 seems to be too difficult to answer in its full generality. For the case $p = 1$, the Maximal Rank Conjecture (MRC)¹ [8] predicts that the multiplication map

$$\text{Sym}^2 H^0(X, L) \xrightarrow{\mu} H^0(X, L^2)$$

is either injective or surjective, or equivalently

$$\min\{k_{1,1}, k_{0,2}\} = 0.$$

Geometrically, this means that the number of quadrics in $\mathbb{P}^r := \mathbb{P}(H^0(L))$ containing X is as simple as the Hilbert function of $X \subset \mathbb{P}^r$ allows.

There are many partial results about the MRC using the so-called “méthode d’Horace” originally proposed by Hirschowitz. We refer to, for instance, [5], [6] for some recent results in this direction.

For higher syzygies, again there are many results (c.f. [1], [2], [4], [7], and [9]). One breakthrough result is Voisin’s solution to the generic Green’s conjecture [13] [14], which answers Question 0.1 for the case $L = K_X$.

Definition 0.2. For $1 \leq p \leq r-1$, we say property $\mathbf{GV}(p)_{g,d}^r$ holds if for general $(X, L) \in \mathcal{W}_{g,d}^r$,

$$\min\{k_{p,1}(X, L), k_{p-1,2}(X, L)\} = 0.$$

Remark 0.3. The MRC implies that property $\mathbf{GV}(1)_{g,d}^r$ always holds provided the Brill-Noether number $\rho := g - (r+1)(g-d+r) \geq 0$. However, property $\mathbf{GV}(p)_{g,d}^r$ does **not** always hold for $p \geq 2$ (c.f Green [10] (4.a.2) for more details).

In this note, we give a sufficient condition (Theorem 1.5) for $\mathbf{GV}(p)_{g,d}^r$ to imply $\mathbf{GV}(p)_{g+1,d+1}^r$. One could use this to set up an inductive argument for the generic vanishing of Koszul cohomology groups. In each step of the induction, r is fixed and g, d go up by 1.

In the case $p = 1$, this sufficient condition turns out to be an surprisingly simple geometric condition on the quadrics containing the first secant variety $\Sigma_1(X)$ of X (Lemma 2.1). We manage to verify this geometric condition and prove

¹In this paper, we will restrict ourselves to only consider quadrics containing X .

Theorem 0.4. The property $\mathbf{GV}(1)_{g,d}^r$ implies $\mathbf{GV}(1)_{g+1,d+1}^r$.

Based on our knowledge about the base cases of the induction, we have

Theorem 0.5. The Maximal Rank Conjecture holds for a general pair $(X, L) \in \mathcal{W}_{g,d}^r$, if $h^1(L) \leq 2$.

An interesting question remaining is that for $p \geq 2$, whether the sufficient condition in Theorem 1.5 has anything to do with higher syzygies of $\Sigma_1(X)$.

1. KOSZUL COHOMOLOGY ON A SINGULAR CURVE

Throughout this section, let $X = Y \cup Z$ be the union of a smooth curve Y of genus g and $Z = \mathbb{P}^1$ meeting at two general points u and v . Consider a line bundle L (up to \mathbb{C}^* -action) on X such that $A := L|_Y$ is a g_d^r and $L|_Z = \mathcal{O}_{\mathbb{P}^1}(1)$. Note that by construction, every section in $H^0(Y, A)$ extends uniquely to a section in $H^0(X, L)$. Thus we have an isomorphism induced by restriction to Y :

$$(1.1) \quad H^0(X, L) \cong H^0(Y, A).$$

Proposition 1.1. Notation as above, if $K_{p,1}(Y, A) = 0$, then $K_{p,1}(X, L) = 0$.

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc} \bigwedge^{p+1} H^0(L) & \longrightarrow & \bigwedge^p H^0(L) \otimes H^0(L) & \longrightarrow & \bigwedge^{p-1} H^0(L) \otimes H^0(L^2) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \\ \bigwedge^{p+1} H^0(A) & \longrightarrow & \bigwedge^p H^0(A) \otimes H^0(A) & \longrightarrow & \bigwedge^{p-1} H^0(A) \otimes H^0(A^2) \end{array}$$

where the vertical arrows are restriction maps to Y . The hypothesis says that the second row is exact in the middle, a simple diagram chasing gives the conclusion. \square

Remark 1.2. The argument in Proposition 1.1 does not generalize to the case $q = 2$ because $H^0(Y, A^2)$ is not isomorphic to $H^0(X, L^2)$.

To study the relation between $K_{p-1,2}(X, L)$ and $K_{p-1,2}(Y, A)$, we use the duality relation [3, p. 21]

$$K_{p-1,2}(Y, A)^\vee \cong K_{r-p,0}(Y, A; K_Y)$$

and compare $K_{r-p,0}(Y, A; K_Y)$ with $K_{r-p,0}(X, L; \omega_X)$. Here ω_X is the dualizing sheaf of X . Its restriction $\omega_X|_Y \cong K_Y(p+q)$ and $\omega_X|_Z \cong \mathcal{O}_{\mathbb{P}^1}$. One checks easily that restriction to Y induces the following isomorphisms:

$$(1.2) \quad H^0(X, \omega_X) \cong H^0(Y, K_Y(u+v)),$$

$$(1.3) \quad H^0(X, \omega_X \otimes L^{-1}) \cong H^0(Y, K_Y \otimes A^{-1}),$$

$$(1.4) \quad H^0(X, \omega_X \otimes L) \cong H^0(Y, K_Y \otimes A(u+v)).$$

Denote M_A the kernel bundle associated to a globally generated line bundle A , defined by the exact sequence

$$0 \rightarrow M_A \rightarrow H^0(Y, A) \otimes \mathcal{O}_Y \xrightarrow{ev} A \rightarrow 0.$$

Taking $(r - p)$ -th wedge product, we obtain

$$0 \longrightarrow \wedge^{r-p} M_A \longrightarrow \wedge^{r-p} H^0(M) \otimes \mathcal{O}_Y \longrightarrow \wedge^{r-p-1} M_A \otimes A \longrightarrow 0.$$

Tensoring the above sequence with K_Y , we obtain an isomorphism [3, Section 2.1]

$$H^0(\wedge^{r-p} M_A \otimes K_Y) \cong \text{Ker}(\delta_0 : \wedge^{r-p} H^0(A) \otimes H^0(K_Y) \longrightarrow \wedge^{r-p-1} H^0(A) \otimes H^0(K_Y \otimes A)),$$

and therefore,

$$(1.5) \quad K_{r-p,0}(Y, A; K_Y) \cong \frac{H^0(\wedge^{r-p} M_A \otimes K_Y)}{\wedge^{r-p+1} H^0(A) \otimes H^0(K_Y \otimes A^{-1})}.$$

Proposition 1.3. We have an isomorphism

$$K_{r-p,0}(X, L; \omega_X) \cong \frac{H^0(\wedge^{r-p} M_A \otimes K_Y(u + v))}{\wedge^{r-p+1} H^0(A) \otimes H^0(K_Y \otimes A^{-1})}.$$

Proof. Consider the following diagram

$$\begin{array}{ccccc} \wedge^{r-p+1} H^0(L) \otimes H^0(\omega_X \otimes L^{-1}) & \xrightarrow{d_{-1}} & \wedge^{r-p} H^0(L) \otimes H^0(\omega_X) & \xrightarrow{d_0} & \wedge^{r-p-1} H^0(L) \otimes H^0(\omega_X \otimes L) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \wedge^{r-p+1} H^0(A) \otimes H^0(K_Y \otimes A^{-1}) & \longrightarrow & \wedge^{r-p} H^0(A) \otimes H^0(K_Y(u + v)) & \xrightarrow{\delta_0} & \wedge^{r-p-1} H^0(A) \otimes H^0(K_Y \otimes A(u + v)), \end{array}$$

where the vertical arrows are induced by restriction to Y . By definition, $K_{r-p,0}(X, L; \omega_X)$ is the cohomology in the middle of the first row. By Equations (1.1) to (1.4), all three vertical arrows are isomorphisms, thus

$$\text{Ker}(d_0) \cong \text{Ker}(\delta_0) \cong H^0(\wedge^{r-p} M_A \otimes K_Y(u + v)).$$

and the statement follows immediately. \square

Corollary 1.4. Notation as above, if

$$h^0(\wedge^{r-p} M_A \otimes K_Y) = h^0(\wedge^{r-p} M_A \otimes K_Y(u + v)),$$

or equivalently,

$$(1.6) \quad h^0((\wedge^p M_A \otimes A(-u - v))) = h^0(\wedge^p M_A \otimes A) - 2 \binom{r}{p},$$

then

$$K_{r-p,0}(X, L; \omega_X) \cong K_{r-p,0}(Y, A; K_Y).$$

Proof. Immediate. The equivalence of the two assumptions followed from Riemann-Roch and the fact that $\wedge^{r-p} M_A^\vee \cong \wedge^p M_A \otimes A$. \square

By degenerating to the pair (X, L) , we obtain

Theorem 1.5. Suppose a general pair (Y, A) in $\mathcal{W}_{g,d}^r$ satisfies one of the two conditions:

- (1) $K_{p,1}(Y, A) = 0$;
- (2) $K_{p-1,2}(Y, A) = 0$ and the vector bundle $\wedge^p M_A \otimes A$ satisfies (1.6) for some $u, v \in Y$.

Then the property $\mathbf{GV}(p)_{g+1,d+1}^r$ holds.

2. THE CASE $p = 1$

In the case $p = 1$, Equation (1.6) has a very geometric interpretation.

Lemma 2.1. For a pair $Y \xrightarrow{\phi|_A} \mathbb{P}^r$ in $\mathcal{W}_{g,d}^r$, the vector bundle $M_A \otimes A$ satisfies equation (1.6) for some $u, v \in Y$ if and only if there exists a quadric hypersurface $Q \subset \mathbb{P}^r$ containing Y but **not** containing its first secant variety $\Sigma_1(Y)$.

Proof. (\Leftarrow) The " \geq " direction of (1.6) is automatically satisfied. For the other direction, consider the following diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(M_A \otimes A(-u-v)) & \longrightarrow & H^0(A) \otimes H^0(A(-u-v)) & \xrightarrow{\mu'} & H^0(A^2(-u-v)) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^0(M_A \otimes A) & \longrightarrow & H^0(A) \otimes H^0(A) & \xrightarrow{\mu} & H^0(A^2). \end{array}$$

We need to show

$$\dim_{\mathbb{C}} \text{Ker}(\mu') \leq \dim_{\mathbb{C}} \text{Ker}(\mu) - 2r.$$

Denote $H_{u,v} := H^0(A) \otimes H^0(A(-u-v))$ and $\overline{H}_{u,v}$ be its image in

$$\frac{H^0(A) \otimes H^0(A)}{\wedge^2 H^0(A)} \cong S^2 H^0(A).$$

Note that

$$\overline{H}_{u,v} \cong \frac{H_{u,v}}{H_{u,v} \cap \wedge^2 H^0(A)},$$

is the space of quadrics which contain the secant line \overline{uv} . Thus

$$\dim_{\mathbb{C}} \overline{H}_{u,v} = \binom{r+2}{2} - 3.$$

We claim that

$$H_{u,v} \cap \wedge^2 H^0(A) = \wedge^2 H^0(A(-u-v)).$$

This is because

$$\begin{aligned} & \dim_{\mathbb{C}} H_{u,v} \cap \wedge^2 H^0(A) \\ &= \dim_{\mathbb{C}} H_{u,v} - \dim_{\mathbb{C}} \overline{H}_{u,v} \\ &= (r+1)(r-1) - [\binom{r+2}{2} - 3] = \binom{r-1}{2} \\ &= \dim_{\mathbb{C}} \wedge^2 H^0(A(-u-v)). \end{aligned}$$

The claim is proved.

By hypothesis, $\overline{\text{Ker}(\mu)} \not\subseteq \overline{H}_{u,v}$ for some $u, v \in Y$ (since $Q \notin \overline{H}_{u,v}$), then it follows that

$$\dim_{\mathbb{C}}(\overline{\text{Ker}(\mu')}) = \dim_{\mathbb{C}}(\overline{\text{Ker}(\mu)} \cap \overline{H}_{u,v}) \leq \dim_{\mathbb{C}}(\overline{\text{Ker}(\mu)} \cap \overline{H}_{u,v}) \leq \dim_{\mathbb{C}}(\overline{\text{Ker}(\mu)}) - 1 =: m - 1.$$

Thus

$$\begin{aligned}
\dim_{\mathbb{C}}(\text{Ker}(\mu')) &\leq m - 1 + \dim_{\mathbb{C}}(\wedge^2 H^0(A) \cap H_{u,v}) \\
&= m - 1 + \dim_{\mathbb{C}}(\wedge^2 H^0(A(-u - v))) \\
&= m - 1 + \binom{r-1}{2} = m + \binom{r+1}{2} - 2r \\
&= \dim_{\mathbb{C}}(\text{Ker}(\mu)) - 2r.
\end{aligned}$$

(\implies) Reverse the above argument we get the “only if” part. □

Lemma 2.2. Suppose $Y \hookrightarrow \mathbb{P}^r$ is a nondegenerate curve in \mathbb{P}^r , then there does **not** exist any quadric hypersurface Q containing $\Sigma_1(Y)$.

Proof. Suppose $Y \subset \Sigma_1(Y) \subset Q$ for some quadric Q . Fix a point $u \in Y$, since Q contains $\Sigma_1(Y)$, Q must contain the variety $\mathcal{J}(u, Y)$ of lines joining u and Y . Thus the quadric Q is singular at u . (If Q is smooth at u , a secant line $\overline{uw} \subset Q$ if and only if $\overline{uw} \subset T_u Q$. Choose $w \in Y \setminus T_u Q$, we have $\overline{uw} \subset \mathcal{J}(u, Y) \subset \Sigma_1(Y)$ but $\overline{uw} \not\subset Q$.) Since u is chosen arbitrarily, we conclude that Q is singular along Y . This is absurd since the singular locus of a quadric is a linear subspace in \mathbb{P}^r which can not contain the nondegenerate curve Y . □

Proof. of **Theorem 0.4.** Follows immediately from Theorem 1.5 and Lemmas 2.1, 2.2. □

3. APPLICATIONS TO THE MAXIMAL RANK CONJECTURE

As an application of Theorem 0.4, we obtain a proof of Theorem 0.5.

We say a triple (g, r, d) , or equivalently $(g, r, h^1 = g - d + r)$ is a base case for the MRC if the Brill-Noether number $\rho := g - (r + 1)h^1 = 0$.

Theorem 3.1. If the MRC holds for all $\rho = 0$ cases, then it holds for arbitrary $\rho \geq 0$ case.

Proof. Apply Theorem 0.4 and induction. Start with any base case, for which we assume property $\mathbf{GV}(1)_{g,d}^r$ holds. In each step of the induction, r and h^1 is fixed and g (equivalently ρ or d) goes up by 1. □

The MRC for the base cases are known to be true when $h^1 \leq 2$. According to the value of h^1 , we have the following.

- (1) $h^1 = 0$. We have $g = 0$ and $d = r \geq 1$, i.e. $(Y, A) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$. The rational normal curves are projectively normal.
- (2) $h^1 = 1$. In this case, $g = r + 1, d = 2r$, i.e. $(Y, A) = (Y, K_Y)$. By Nother's theorem, canonical curves are projectively normal ($r \geq 2$).
- (3) $h^1 = 2$. Then $g = 2r + 2, d = 3r$. Such pairs (Y, A) are projectively normal for $r \geq 4$ is the main result of [11] (The MRC is easy to check when $r = 2$ or 3).

Farkas [9] also proved that $\mathbf{GV}(1)_{s(2s+1), 2s(s+1)}^{2s}$ holds for any $s \geq 1$. This covers the base cases when $h^1 = s$ and $r = 2s$.

Proof. of **Theorem 0.5.** Follows immediately from the base cases with $h^1 \leq 2$ and Theorem 3.1. \square

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